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Effective action of $\mathcal{N}=4$ super Yang-Mills: $\mathcal{N}=2$ superspace approach

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Abstract

Using the $\mathcal{N}=2$ off-shell formulation in harmonic superspace for $\mathcal{N}=4$ super Yang-Mills theory, we present a representation of the one-loop effective action which is free of so-called coinciding harmonic singularities and admits a straightforward evaluation of low-energy quantum corrections in the framework of an $\mathcal{N}=2$ superfield heat kernel technique. We illustrate our approach by computing the low-energy effective action on the Coulomb branch of $SU(2)$ $\mathcal{N}=4$ super Yang-Mills. Our work provides the first derivation of the low-energy action of $\mathcal{N}=4$ super Yang-Mills theory directly in $\mathcal{N}=2$ superspace without any reduction to $\mathcal{N}=1$ superfields and for a generic background $\mathcal{N}=2$ Yang-Mills multiplet.

1 Introduction

Harmonic superspace [1, 2] is a manifestly supersymmetric construction which allows the description of general $\mathcal{N} = 2$ super Yang-Mills theories in terms of unconstrained superfields whilst preserving the $SU(2)_R$ automorphism symmetry of the $\mathcal{N} = 2$ Poincaré superalgebra. The usual superspace variables are supplemented by harmonics on an internal S^2. The Feynman rules in this approach were elaborated early in the history of harmonic superspace [3]. Less well-developed is the background field formulation in harmonic superspace [4, 5], which is expected to be vital for computing low-energy effective actions in quantum $\mathcal{N} = 2$ super Yang-Mills theories (the background field formulation of [4, 5] has already been used to prove the $\mathcal{N} = 2$ non-renormalization theorem [6] and to compute the leading non-holomorphic quantum corrections in the $\mathcal{N} = 4$ super Yang-Mills theory [5, 7]; see also [8]).

Until now, applications of the background field formulation in harmonic superspace have been incomplete at the one-loop level because of a technical problem. This problem is characteristic of all $\mathcal{N} = 2$ super Yang-Mills theories, but it manifests itself most explicitly in the case of $\mathcal{N} = 4$ SYM, which is a special $\mathcal{N} = 2$ model in which no holomorphic quantum corrections are present. The one-loop effective action of $\mathcal{N} = 4$ SYM consists of two contributions, each of which contains harmonic distributions on the internal S^2 at coinciding points, and hence singularities [5]. Such singularities of harmonic supergraphs were first discussed in [9]. Unlike ultraviolet divergences in quantum field theory, the coinciding harmonic singularities have no physical origin; they can appear only at intermediate stages of calculations and must cancel each other in the final expressions for physical observables. A precise mechanism for the cancellation of coinciding harmonic singularities was found in [5] only for the case when the background $\mathcal{N} = 2$ gauge multiplet satisfies the equations of motion. The use of such a background was sufficient for the authors of [5, 7] to compute the leading non-holomorphic quantum correction $H(W, \bar{W})$ on the Coulomb branch of $\mathcal{N} = 4$ SYM (which was also derived in [10, 11, 12] by different methods).

In the present paper, we present a resolution of the problem of coinciding harmonic singularities in the case of $\mathcal{N} = 4$ SYM and obtain a representation for the one-loop effective action which is suited to computation of the low-energy effective action in the framework of an $\mathcal{N} = 2$ superfield heat kernel technique (similar to the standard $\mathcal{N} = 1$ results [13, 14, 15, 16]). Full details will appear in a following paper [17]. Our work provides the first derivation of the low-energy action of $\mathcal{N} = 4$ SYM directly in $\mathcal{N} = 2$

superspace without any reduction to $\mathcal{N}=1$ superfields and for a generic background $\mathcal{N}=2$ Yang-Mills multiplet.

Our $\mathcal{N}=2$ harmonic superspace conventions correspond to those adopted in ref. [3], apart from the fact that the Greek letter 'ζ' is used to denote all the coordinates of the analytic subspace, with $\mathrm{d}\zeta^{(-4)}$ the analytic subspace measure. The algebra of $\mathcal{N}=2$ gauge covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\alpha}_j)$ was established in [18]; the basic anti-commutation relations can be expressed

$$
\begin{gathered}
\{\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta\} = \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^+_{\dot\beta}\} = \{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot\beta}\} = 0\ , \\
\{\bar{\mathcal{D}}^+_{\dot\alpha}, \mathcal{D}^-_\alpha\} = -\{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^-_{\dot\alpha}\} = 2\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\ , \\
\{\mathcal{D}^+_\alpha, \mathcal{D}^-_\beta\} = -2\mathrm{i}\,\varepsilon_{\alpha\beta}\bar{\mathcal{W}}\ , \qquad \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^-_{\dot\beta}\} = 2\mathrm{i}\,\varepsilon_{\dot\alpha\dot\beta}\mathcal{W}\ ,
\end{gathered}
\tag{1.1}
$$

where $\mathcal{D}^\pm_\alpha = \mathcal{D}^i_\alpha\, u^\pm_i$, $\bar{\mathcal{D}}^\pm_{\dot\alpha} = \bar{\mathcal{D}}^i_{\dot\alpha}\, u^\pm_i$, with u^+_i and u^-_i the harmonics [1]. The field strength $\mathcal{W}$ satisfies the Bianchi identities [18]

$$
\bar{\mathcal{D}}^i_{\dot\alpha}\,\mathcal{W} = 0\ , \qquad \mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_\alpha\,\mathcal{W} = \bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar{\mathcal{D}}^{j)\dot\alpha}\,\bar{\mathcal{W}} = 0\ . \tag{1.2}
$$

In harmonic superspace, a full set of gauge covariant derivatives includes the spherical derivatives ($\mathcal{D}^{++}$, $\mathcal{D}^{--}$, $\mathcal{D}^0$), which form the algebra $su(2)$ and satisfy special commutation relations with $\mathcal{D}^\pm_\alpha$ and $\bar{\mathcal{D}}^\pm_{\dot\alpha}$ [3]. Here, $\mathcal{D}^0$ is the harmonic $U(1)$ charge operator, and the important property of $\mathcal{D}^{++}$ is that it preserves analyticity. Throughout this paper, we use the so-called τ-frame [1, 3], and the adjoint representation of the gauge group is assumed.

2 The one-loop effective action

A formal definition of the one-loop effective action for $\mathcal{N}=4$ super Yang-Mills theory has been given in ref. [5]:

$$
\Gamma^{(1)} = \frac{\mathrm{i}}{2}\,\mathrm{Tr}_{\,(2,2)}\,\ln\widehat{\Box} - \frac{\mathrm{i}}{2}\,\mathrm{Tr}_{\,(0,4)}\,\ln\widehat{\Box}\ , \tag{2.1}
$$

where $\widehat{\Box}$ is the analytic d'Alembertian

$$
\begin{aligned}
\widehat{\Box} = {}& \mathcal{D}^m\mathcal{D}_m + \frac{\mathrm{i}}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}^-_\alpha + \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^+_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha} - \frac{\mathrm{i}}{4}(\mathcal{D}^{+\alpha}\mathcal{D}^+_\alpha\mathcal{W})\mathcal{D}^{--} \\
& + \frac{\mathrm{i}}{8}[\mathcal{D}^{+\alpha}, \mathcal{D}^-_\alpha]\mathcal{W} + \frac{1}{2}\{\bar{\mathcal{W}}, \mathcal{W}\}
\end{aligned}
\tag{2.2}
$$

possessing the following important properties

$$[\mathcal{D}^+_\alpha\,,\,\widehat{\Box}] = 0\ , \qquad [\bar{\mathcal{D}}^+_{\dot\alpha}\,,\,\widehat{\Box}] = 0\ . \tag{2.3}$$

The traces are functional traces of operators acting on analytic superfields of appropriate $U(1)$ charge. Specifically, if $\mathcal{F}^{(p,4-p)}(\zeta_1,\zeta_2)$ is the kernel of an operator acting on the space of covariantly analytic superfields of $U(1)$ charge p, then

$$\mathrm{Tr}\ \mathcal{F}^{(p,4-p)} = \mathrm{tr}\int \mathrm{d}\zeta^{(-4)}\ \mathcal{F}^{(p,4-p)}(\zeta,\zeta)\ ,$$

where the trace 'tr' is over group indices. The two contributions to $\Gamma^{(1)}$ can be represented by path integrals over unconstrained analytic superfields $u^{++},\ v^{++}$ and $\rho^{(+4)},\ \sigma$ as follows:

$$\Big(\mathrm{Det}_{(2,2)}\,\widehat{\Box}\Big)^{-1} = \int[\mathrm{d}u^{++}]\,[\mathrm{d}v^{++}]\ \exp\Big\{\mathrm{tr}\int \mathrm{d}\zeta^{(-4)}\ u^{++}\widehat{\Box}\,v^{++}\Big\}\ , \tag{2.4}$$

$$\Big(\mathrm{Det}_{(0,4)}\,\widehat{\Box}\Big)^{-1} = \int[\mathrm{d}\rho^{(+4)}]\,[\mathrm{d}\sigma]\,\exp\Big\{\mathrm{tr}\int \mathrm{d}\zeta^{(-4)}\ \rho^{(+4)}\widehat{\Box}\,\sigma\Big\}\ . \tag{2.5}$$

Both terms in (2.1) contain harmonic singularities due to the coincidence limit involved in the functional traces. However, these cancel each other, and we claim that the resulting well-defined expression for the one-loop effective action is

$$\Gamma^{(1)} = -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\ \mathrm{Tr}\ \Big(\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}}\ \Pi^{(2,2)}_{\mathrm{T}}\Big)\ . \tag{2.6}$$

Here, $\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2)$ is a projector on the space of covariantly analytic transverse superfields $\mathcal{G}^{++}(\zeta)$ defined by the constraints

$$\mathcal{D}^{+\hat\alpha}\mathcal{G}^{++} = 0\ , \qquad \mathcal{D}^{++}\mathcal{G}^{++} = 0\ , \tag{2.7}$$

where $\mathcal{D}^{+\hat\alpha} = (\mathcal{D}^{+\alpha}, \bar{\mathcal{D}}^{+\dot\alpha})$. The properties of $\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2)$ are:

$$\mathcal{D}^{+\hat\alpha}_1\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2) = \mathcal{D}^{+\hat\alpha}_2\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2) = 0\ , \tag{2.8}$$

$$\mathcal{D}^{++}_1\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2) = \mathcal{D}^{++}_2\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2) = 0\ , \tag{2.9}$$

$$\int \mathrm{d}\zeta_3^{(-4)}\ \Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_3)\,\Pi^{(2,2)}_{\mathrm{T}}(\zeta_3,\zeta_2) = \Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2)\ , \tag{2.10}$$

$$\Big(\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2)\Big)^{\mathrm{T}} = \Pi^{(2,2)}_{\mathrm{T}}(\zeta_2,\zeta_1)\ . \tag{2.11}$$

The projector $\Pi^{(2,2)}_{\mathrm{T}}$ is related to the Green's function $G^{(0,0)}(\zeta_1,\zeta_2)$ for a ω–hypermultiplet coupled to background $\mathcal{N}=2$ gauge superfields, which satisfies the equation

$$(\mathcal{D}^{++}_1)^2 G^{(0,0)}(\zeta_1,\zeta_2) = -\delta_A^{(4,0)}(\zeta_1,\zeta_2). \tag{2.12}$$

This Green's function can be expressed explicitly [3, 6] in the form[1]

$$G^{(0,0)}(\zeta_1,\zeta_2) = \frac{1}{\widehat{\Box}_1}(\mathcal{D}_1^+)^4\,(\mathcal{D}_2^+)^4\left\{\delta^{12}(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}\right\}\ , \tag{2.13}$$

and is manifestly analytic in both arguments. Less obvious is the fact that $G^{(0,0)}(\zeta_1,\zeta_2)$ is also symmetric,

$$\Big(G^{(0,0)}(\zeta_1,\zeta_2)\Big)^{\mathrm{T}} = G^{(0,0)}(\zeta_2,\zeta_1)\ . \tag{2.14}$$

The latter property will be discussed in detail in [17]. It follows from eq. (2.12) that the analytic two-point function

$$\Pi_{\mathrm{L}}^{(2,2)}(\zeta_1,\zeta_2) = \mathcal{D}_1^{++}\mathcal{D}_2^{++}G^{(0,0)}(\zeta_1,\zeta_2) \tag{2.15}$$

has the properties

$$\int \mathrm{d}\zeta_3^{(-4)}\,\Pi_{\mathrm{L}}^{(2,2)}(\zeta_1,\zeta_3)\,\Pi_{\mathrm{L}}^{(2,2)}(\zeta_3,\zeta_2) = \Pi_{\mathrm{L}}^{(2,2)}(\zeta_1,\zeta_2)\ , \tag{2.16}$$

$$\mathcal{D}_1^{++}\Pi_{\mathrm{L}}^{(2,2)}(\zeta_1,\zeta_2) = \mathcal{D}_1^{++}\delta_A^{(2,2)}(\zeta_1,\zeta_2)\ . \tag{2.17}$$

Therefore, $\Pi_{\mathrm{L}}^{(2,2)}$ is the projector on the space of longitudinal analytic superfields

$$\mathcal{F}^{++} = \mathcal{D}^{++}\Lambda\ , \qquad \mathcal{D}^{+\hat{\alpha}}\Lambda = 0\ . \tag{2.18}$$

As a result, $\Pi_{\mathrm{T}}^{(2,2)}$ can be expressed as

$$\Pi_{\mathrm{T}}^{(2,2)}(\zeta_1,\zeta_2) = \delta_A^{(2,2)}(\zeta_1,\zeta_2)\ -\ \Pi_{\mathrm{L}}^{(2,2)}(\zeta_1,\zeta_2)\ , \tag{2.19}$$

which establishes the connection with the Green's function $G^{(0,0)}(\zeta_1,\zeta_2)$. When the $\mathcal{N}=2$ vector multiplet is set to zero, $\Pi_{\mathrm{T}}^{(2,2)}$ reduces to the flat projector derived in [3].

The representation (2.6) for the one-loop effective action can readily be deduced from the formal one given by eq. (2.1) in the case when the background gauge superfield satisfies the classical equation of motion

$$\mathcal{D}^{\alpha(i}\mathcal{D}_\alpha^{j)}\mathcal{W} = 0 \qquad \Longleftrightarrow \qquad [\mathcal{D}^{++},\widehat{\Box}] = 0\ . \tag{2.20}$$

In this case, one can argue as follows:

$$\begin{aligned}
\Gamma^{(1)} &= -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\ \mathrm{Tr}\ \Big\{\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\ \delta_A^{(2,2)}(\zeta_1,\zeta_2) - \mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\ \delta_A^{(0,4)}(\zeta_1,\zeta_2)\Big\} \\
&= -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\ \mathrm{Tr}\ \Big\{\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\ \delta_A^{(2,2)}(\zeta_1,\zeta_2) + \mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\ (\mathcal{D}_2^{++})^2G^{(0,0)}(\zeta_1,\zeta_2)\Big\} \\
&= -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\ \mathrm{Tr}\ \Big\{\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\ \delta_A^{(2,2)}(\zeta_1,\zeta_2) - \mathcal{D}_1^{++}\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\ \mathcal{D}_2^{++}G^{(0,0)}(\zeta_1,\zeta_2)\Big\} \\
&= -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\ \mathrm{Tr}\ \Big\{\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\Big(\delta_A^{(2,2)}(\zeta_1,\zeta_2) - \Pi_{\mathrm{L}}^{(2,2)}(\zeta_1,\zeta_2)\Big)\Big\}\ .
\end{aligned} \tag{2.21}$$

[1]Here $(\mathcal{D}^+)^4 = \frac{1}{16}\mathcal{D}^{+\alpha}\mathcal{D}_\alpha^+\,\bar{\mathcal{D}}_{\dot\alpha}^+\bar{\mathcal{D}}^{+\dot\alpha}$.

The cyclic property of the functional trace has been used in going from the second line to the third line.

In the off-shell case, the operators $\mathcal{D}^{++}$ and $\widehat{\Box}$ do not commute in general[2]:

$$[\mathcal{D}^{++},\widehat{\Box}]\Phi^{(q)} = \frac{\mathrm{i}}{4}\,(1-q)(\mathcal{D}^+\mathcal{D}^+\mathcal{W})\,\Phi^{(q)} \tag{2.22}$$

for an arbitrary analytic superfield $\Phi^{(q)}$ with $U(1)$ charge q. Thus, the process of interchanging the operators $\mathcal{D}^{++}_1$ and $\exp(-\mathrm{i}\,t\,\widehat{\Box}_1)$ in going from the third line of (2.21) to the fourth line introduces an extra contribution to the effective action containing factors of the classical equation of motion. As is well known, contributions to the effective action containing factors of the classical equations of motion are ambiguous and may be ignored. On the other hand, this extra contribution proves to involve coinciding harmonic singularities which must be regularized; for a regularization preserving analyticity, the whole extra contribution can be shown to vanish.

It follows from (2.9) that $\Pi^{(2,2)}_{\rm T}$ has the following dependence on the harmonics:

$$\Pi^{(2,2)}_{\rm T}(z_1,u_1\,,\,z_2,u_2) = \Pi^{(ij)(kl)}_{\rm T}(z_1\,,\,z_2)\;(u^+_1)_i(u^+_1)_j\,(u^+_2)_k(u^+_2)_l\;. \tag{2.23}$$

Therefore the effective action (2.6) is by construction free of harmonic singularities. We will present a representation for $\Pi^{(2,2)}_{\rm T}$ which is useful for heat kernel calculations. Using the standard properties of harmonic distributions [3] and the identity

$$[\mathcal{D}^{++},\frac{1}{\widehat{\Box}}] = -\frac{1}{\widehat{\Box}}\;[\mathcal{D}^{++},\widehat{\Box}]\;\frac{1}{\widehat{\Box}}\;, \tag{2.24}$$

we deduce from the definition (2.19) of $\Pi^{(2,2)}_{\rm T}$ that

$$\begin{aligned}\Pi^{(2,2)}_{\rm T}(\zeta_1,\zeta_2) = {}& \frac{\mathrm{i}}{4}\,\frac{1}{\widehat{\Box}_1}\,(\mathcal{D}^+_1\mathcal{D}^+_1\mathcal{W}_1)\,\frac{1}{\widehat{\Box}_1}\,(\mathcal{D}^+_1)^4(\mathcal{D}^+_2)^4\,\delta^{12}(z_1-z_2)\,\frac{(u^-_1u^+_2)}{(u^+_1u^+_2)^3}\\ &-\frac{1}{\widehat{\Box}_1}\,(\mathcal{D}^+_1)^4(\mathcal{D}^+_2)^4\,\delta^{12}(z_1-z_2)\,\frac{1}{(u^+_1u^+_2)^2}\;.\end{aligned} \tag{2.25}$$

The next step is to realize that the covariant derivatives $\mathcal{D}^{+\hat\alpha}_2$ in a two-point function of the form

$$(\mathcal{D}^+_1)^4(\mathcal{D}^+_2)^4\,\delta^{12}(z_1-z_2)\,\frac{1}{(u^+_1u^+_2)^q}\;, \tag{2.26}$$

can be re-expressed in terms of the covariant derivatives $\mathcal{D}^{\pm\hat\alpha}_1$ with the help of the identity

$$\Psi^+_2 = (u^+_1u^+_2)\,\Psi^-_1 - (u^-_1u^+_2)\,\Psi^+_1\;, \qquad \Psi^\pm = \Psi^i\,u^\pm_i \tag{2.27}$$

[2]However, from eq. (2.22) one derives $[\mathcal{D}^{++},\widehat{\Box}]\,q^+ = 0$ and $[(\mathcal{D}^{++})^2,\widehat{\Box}]\,\omega = 0$.

along with the algebra of gauge covariant derivatives which implies, in particular, that

$$(\mathcal{D}^+)^4\, \mathcal{D}^+_{\dot\alpha} = \mathcal{D}^+_{\dot\alpha}\, (\mathcal{D}^+)^4 = 0\ . \tag{2.28}$$

The result is

$$\begin{aligned}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\, \frac{\delta^{12}(z_1-z_2)}{(u_1^+u_2^+)^q} &= (\mathcal{D}_1^+)^4 \left\{ (\mathcal{D}_1^-)^4\, \frac{1}{(u_1^+u_2^+)^{q-4}} - \frac{\mathrm{i}}{2}\,\Delta_1^{--}\, \frac{(u_1^-u_2^+)}{(u_1^+u_2^+)^{q-3}} \right.\\ &\left. -\widehat{\Box}_1\, \frac{(u_1^-u_2^+)^2}{(u_1^+u_2^+)^{q-2}} + \frac{\mathrm{i}}{4}(q-3)\,(\mathcal{D}_1^+\mathcal{D}_1^+\mathcal{W}_1)\, \frac{(u_1^-u_2^+)^3}{(u_1^+u_2^+)^{q-1}} \right\} \delta^{12}(z_1-z_2)\ , \end{aligned} \tag{2.29}$$

where

$$\begin{aligned}\Delta^{--} = \mathcal{D}^{\alpha\dot\alpha}\mathcal{D}^-_\alpha\bar{\mathcal{D}}^-_{\dot\alpha} + \frac{1}{2}\mathcal{W}(\mathcal{D}^-)^2 + \frac{1}{2}\bar{\mathcal{W}}(\bar{\mathcal{D}}^-)^2 \\ + (\mathcal{D}^-\mathcal{W})\mathcal{D}^- + (\bar{\mathcal{D}}^-\bar{\mathcal{W}})\bar{\mathcal{D}}^- + \frac{1}{2}(\mathcal{D}^-\mathcal{D}^-\mathcal{W})\ .\end{aligned} \tag{2.30}$$

It is worth noting that eq. (2.29) simplifies in the two special cases $q=2$ and $q=3$, which are of importance here. Putting all these results together, we obtain the following representation for $\Pi^{(2,2)}_{\mathrm{T}}$:

$$\begin{aligned}\Pi^{(2,2)}_{\mathrm{T}}(\zeta_1,\zeta_2) &= (\mathcal{D}_1^+)^4\, \delta^{12}(z_1-z_2)\, (u_1^-u_2^+)^2 \\ &- \frac{1}{\widehat{\Box}_1}\left\{ (u_1^+u_2^+) - \frac{\mathrm{i}}{4}\,(\mathcal{D}_1^+\mathcal{D}_1^+\mathcal{W}_1)\,\frac{1}{\widehat{\Box}_1}\,(u_1^-u_2^+)\right\} \\ &\times (\mathcal{D}_1^+)^4 \left\{ (u_1^+u_2^+)\,(\mathcal{D}_1^-)^4 - \frac{\mathrm{i}}{2}(u_1^-u_2^+)\,\Delta_1^{--}\right\} \delta^{12}(z_1-z_2)\ .\end{aligned} \tag{2.31}$$

The factors of $1/\widehat{\Box}$ are to be understood in the sense of Schwinger's proper-time representation,

$$\frac{1}{\widehat{\Box}} = \mathrm{i}\int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\mathrm{i}\,s\,\widehat{\Box}}\ . \tag{2.32}$$

It is also important to note that the covariant transverse projector $\Pi^{(2,2)}_{\mathrm{T}}$ has the property

$$\mathrm{Tr}\ \Pi^{(2,2)}_{\mathrm{T}}\ =\ 0\ . \tag{2.33}$$

Using this expression for $\Pi^{(2,2)}_{\mathrm{T}}$, the following key features of the representation (2.6) for the one-loop effective action are apparent:

1. Eq. (2.6) is free of coinciding harmonic singularities;

2. The representation (2.6) provides a simple and powerful scheme for computing the effective action in the framework of an $\mathcal{N}=2$ superfield proper-time technique;

3. Unlike the approach of ref. [5], the representation (2.6) is valid for arbitrary background $\mathcal{N}=2$ gauge superfields.

Using eq. (2.9) and rewriting (2.6) in the form

$$\Gamma^{(1)} = -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\,\mathrm{Tr}\,\Big(\Pi^{(2,2)}_{\mathrm{T}}\;\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}}\;\Pi^{(2,2)}_{\mathrm{T}}\Big)\;, \tag{2.34}$$

it can be seen that the one-loop effective action takes the form

$$\Gamma^{(1)} = \int \mathrm{d}\zeta^{(-4)}\;\mathcal{L}^{(+4)}_{\mathrm{eff}}[\mathcal{W},\bar{\mathcal{W}}]\;, \qquad \mathcal{D}^+_{\hat{\alpha}}\mathcal{L}^{(+4)}_{\mathrm{eff}} = \mathcal{D}^{++}\mathcal{L}^{(+4)}_{\mathrm{eff}} = 0\;, \tag{2.35}$$

with $\mathcal{L}^{(+4)}_{\mathrm{eff}}$ a gauge invariant functional of the $\mathcal{N}=2$ Yang-Mills multiplet. Without loss of generality, $\mathcal{L}^{(+4)}_{\mathrm{eff}}$ can be expressed in the form

$$\mathcal{L}^{(+4)}_{\mathrm{eff}}[\mathcal{W},\bar{\mathcal{W}}] \;=\; (\mathcal{D}^+)^4\mathcal{L}_{\mathrm{eff}}[\mathcal{W},\bar{\mathcal{W}}]\;, \tag{2.36}$$

with $\mathcal{L}_{\mathrm{eff}}[\mathcal{W},\bar{\mathcal{W}}]$ harmonic-independent, and hence $\Gamma^{(1)}$ can be rewritten as

$$\Gamma^{(1)} = \int \mathrm{d}^{12}z\;\mathcal{L}_{\mathrm{eff}}[\mathcal{W},\bar{\mathcal{W}}]\;. \tag{2.37}$$

More exotic quantum corrections

$$\mathrm{tr}\,\Big\{(\mathcal{D}^+)^2 f(\mathcal{W})\;(\mathcal{D}^+)^2 g(\mathcal{W})\Big\} \quad + \quad \text{c.c.}$$

conflict with superconformal symmetry, although such corrections are not forbidden by the above conditions on $\mathcal{L}^{(+4)}_{\mathrm{eff}}$.

3 On-shell background

If the background gauge multiplet is on-shell, $\mathcal{D}^+\mathcal{D}^+W=0$, the analytic d'Alembertian does not involve any harmonic derivative $\mathcal{D}^{--}$ and eq. (2.31) takes the simpler form

$$\begin{aligned}\Pi^{(2,2)}_{\mathrm{T}}(1,2) &= (\mathcal{D}^+_1)^4\;\delta^{12}(z_1-z_2)\;(u^-_1u^+_2)^2 \\ &\quad - (u^+_1u^+_2)\;\frac{1}{\widehat{\Box}_1}(\mathcal{D}^+_1)^4\Big\{(u^+_1u^+_2)\;(\mathcal{D}^-_1)^4 - \frac{\mathrm{i}}{2}(u^-_1u^+_2)\;\Delta^{--}_1\Big\}\;\delta^{12}(z_1-z_2)\;.\end{aligned} \tag{3.1}$$

Using this and the identities

$$(u^-_1u^+_2)|_{1=2} = -1\;, \qquad (u^+_1u^+_2)|_{1=2} = 0, \tag{3.2}$$

one immediately observes that the structure of the effective action (2.6) drastically simplifies to

$$\Gamma^{(1)} = -\frac{\mathrm{i}}{2}\int_0^\infty \frac{\mathrm{d}t}{t} \int \mathrm{d}\zeta_1^{(-4)}\ \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}\,t\,\widehat{\Box}_1}\ (\mathcal{D}_1^+)^4\ \delta^{12}(z_1 - z_2)\right)\Big|_{z_1=z_2} . \tag{3.3}$$

The representation (3.3) allows us to compute $\Gamma^{(1)}$ in a manner almost identical to $\mathcal{N}=1$ superfield heat kernel calculations [14, 15]. This is facilitated by the similarity of the on-shell $\mathcal{N}=2$ analytic d'Alembertian

$$\widehat{\Box} = \mathcal{D}^m\mathcal{D}_m + \frac{\mathrm{i}}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}^-_\alpha + \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^+_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha} + \frac{1}{2}\{\bar{\mathcal{W}},\mathcal{W}\}$$

to the $\mathcal{N}=1$ full superspace d'Alembertian

$$\widehat{\Box} = \mathcal{D}^m\mathcal{D}_m - \mathcal{W}^\alpha\mathcal{D}_\alpha - \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha},$$

as well as the similarity of the $\mathcal{N}=2$ analytic delta function $(\mathcal{D}_1^+)^4\ \delta^{12}(z_1 - z_2)$ with the $\mathcal{N}=1$ full superspace delta function $\delta^8(z_1 - z_2)$. For simplicity, here we choose the gauge group to be $SU(2)$ and restrict our consideration to the Coulomb branch of the theory,

$$[\mathcal{W}\,,\,\bar{\mathcal{W}}]\ =\ 0\ , \tag{3.4}$$

and explicitly $\mathcal{W} = \frac{1}{2}\sigma_3\, W$. On the Coulomb branch, one can use a derivative expansion for the effective action. Ignoring all contributions with vector derivatives of W and $\bar W$, direct heat kernel calculations of the effective action (3.3) lead to

$$\begin{aligned} \mathcal{L}^{(+4)}_{\text{eff}} &= (D^+)^4\mathcal{L}_{\text{eff}}\ , \\ \mathcal{L}_{\text{eff}} &= \frac{1}{16\pi^2}\ln\frac{W}{\mu}\,\ln\frac{\bar W}{\mu} + \frac{1}{8\pi^2}\int_0^\infty \mathrm{d}t\, t\,\mathrm{e}^{-t}\ \Omega(t\Psi\,,\,t\bar\Psi)\ , \end{aligned} \tag{3.5}$$

where

$$\bar\Psi^2 = \frac{1}{\bar W^2}\,D^4\,\ln\frac{W}{\mu}\ , \qquad \Psi^2 = \frac{1}{W^2}\,\bar D^4\,\ln\frac{\bar W}{\mu}\ , \tag{3.6}$$

and $\Omega(x,y) = \Omega(y,x)$ is the analytic function related to

$$\omega(x,y) = \omega(y,x) = \frac{\cosh x - 1}{x^2}\,\frac{\cosh y - 1}{y^2}\,\frac{x^2 - y^2}{\cosh x - \cosh y} - \frac{1}{2} \tag{3.7}$$

by the following rule: if

$$\omega(x,y) = \sum_{m,n=1}^{\infty} c_{m,n}\, x^{2m}\, y^{2n}\ , \tag{3.8}$$

then

$$\Omega(x,y) = \frac{1}{4} \sum_{m,n=1}^{\infty} \frac{c_{m,n}}{m(2m+1)n(2n+1)} \, x^{2m} \, y^{2n} \ . \tag{3.9}$$

In (3.5), μ is a formal scale which is introduced to make the argument of the logarithm dimensionless. It drops out from the effective action and the superconformal scalars (3.6). The one-loop effective Lagrangian (3.5) coincides with the one found in [16] on the basis of $\mathcal{N}=1$ heat kernel calculations and $\mathcal{N}=2$ superconformal considerations.

4 Off-shell background

For a generic off-shell background superfield, we have to take into account numerous terms containing factors of the classical equation of motion. The main role of such additional terms is to combine with the structures present in the on-shell case to make $\mathcal{L}^{(+4)}_{\rm eff}$ analytic. We illustrate this statement using the example of the leading term in (3.5), which is quartic in spinor derivatives:

$$\begin{aligned} \frac{1}{(4\pi)^2} \, (D^+)^4 \, \ln \frac{W}{\mu} \, \ln \frac{\bar W}{\mu} = \frac{1}{(16\pi)^2} \, \Big\{ & \frac{D^+W \, D^+W}{W^2} \, \frac{\bar D^+ \bar W \, \bar D^+ \bar W}{\bar W^2} \\ & + \frac{D^+D^+W}{W} \, \frac{\bar D^+ \bar D^+ \bar W}{\bar W} \\ & - \frac{D^+D^+W}{W} \, \frac{\bar D^+ \bar W \, \bar D^+ \bar W}{\bar W^2} - \frac{D^+W \, D^+W}{W^2} \frac{\bar D^+ \bar D^+ \bar W}{\bar W} \Big\} \ . \end{aligned} \tag{4.1}$$

Only the first term on the right-hand side is present in the on-shell case. It comes from evaluating the effective action with only the first term in (2.31) taken into account, in which case (2.6) yields (with the integral over the analytic subspace omitted):

$$\begin{aligned} & -{\rm i} \int_0^\infty \frac{{\rm d}t}{t} \, {\rm e}^{-{\rm i}\,t\,\widehat{\Box}} \, (\mathcal{D}^+)^4 \, \delta^{12}(z-z') \Big|_{z=z'} \\ &= -{\rm i} \int_0^\infty \frac{{\rm d}t}{t} \, \frac{t^4}{4} \, D^+W \, D^+W \bar D^+ \bar W \, \bar D^+ \bar W \, {\rm e}^{-{\rm i}\,t\,(W\bar W - {\rm i}\varepsilon)} \, {\rm e}^{-{\rm i}\,t\,\Box} \, \delta^4(x-x') \Big|_{z=z'} + \cdots \\ &= \frac{1}{(16\pi)^2} \, D^+W \, D^+W \bar D^+ \bar W \, \bar D^+ \bar W \int_0^\infty {\rm d}t \, t \, {\rm e}^{-t\,W\bar W} \ + \cdots , \end{aligned} \tag{4.2}$$

where we have used the identity

$$(\mathcal{D}^-)^4 \, (\mathcal{D}^+)^4 \, \delta^8(\theta-\theta') \Big|_{\theta=\theta'} = 1 \tag{4.3}$$

and applied Schwinger's rotation $t \to -{\rm i}\,t$ of the proper-time parameter. The second term in (4.1) originates from taking into account the contribution in (2.31) which is

proportional to $(\mathcal{D}_1^+)^4\,(\mathcal{D}_1^-)^4$; here the operator $\mathcal{D}^{--}$ appearing in $\widehat{\Box}$ must hit factor(s) of $(u_1^+ u_2^+)$ in order to get a non-vanishing contribution in the limit $u_1 = u_2$. Using $\widehat{\Box}^{-2} = \int_0^\infty \mathrm{d}s\; s\; \mathrm{e}^{-\mathrm{i}\, s\, \widehat{\Box}}$, one obtains

$$
\begin{aligned}
&-\frac{1}{4}(D^+D^+W)\int_0^\infty \frac{\mathrm{d}t}{t}\int_0^\infty \mathrm{d}s\; s\; \mathrm{e}^{-\mathrm{i}\,(t+s)\,\widehat{\Box}}\;(u^+u'^+)(u^-u'^+)\delta^{12}(z-z')\Big|_{z=z',u=u'} \\
&-\int_0^\infty \frac{\mathrm{d}t}{t}\int_0^\infty \mathrm{d}s\; \mathrm{e}^{-\mathrm{i}\,(t+s)\,\widehat{\Box}}\;(u^+u'^+)^2\delta^{12}(z-z')\Big|_{z=z',u=u'} \\
&= \frac{1}{(16\pi)^2}\,(D^+D^+W)^2\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}s\;\frac{\mathrm{e}^{-(t+s)\,W\bar{W}}}{t+s} + \cdots \\
&= \frac{1}{(16\pi)^2}\,\frac{D^+D^+W}{W}\,\frac{\bar{D}^+\bar{D}^+\bar{W}}{\bar{W}} + \cdots .
\end{aligned}
\tag{4.4}
$$

Finally, the terms in the third line of (4.1) emerge when we take into account the structure in (2.31) involving the operator Δ^{--}.

Note added: While preparing this paper, we recognized how to extend the approach advocated here to the case of generic $\mathcal{N}=2$ super Yang-Mills models. The details will be reported in a forthcoming publication [17].

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